Nonequilibrium transitions induced by the cross-correlation of white noises

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We study the role that the cross-correlation of noises plays in the statistical behavior of systems driven by two multiplicative Gaussian white noises. The temporal evolution of the system is described by a Langevin equation, for which we adopt a general interpretation that includes the Ito as well as the Stratonovich interpretation. We derive the stochastically equivalent Fokker-Planck equation by means of the two-stage averaging of a state-dependent function. Analyzing the stationary solution of the Fokker-Planck equation for specific examples, we show explicitly that the cross-correlation of white noises can induce nonequilibrium transitions.

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I. INTRODUCTION

A large variety of phenomena in physics, chemistry, biology, and other fields involve the interaction of a nonlinear system with a fluctuating environment. The latter is often modeled as a source of external noise with known statistical characteristics. Most commonly, the external noise is considered to be Gaussian and white, since this provides a satisfactory idealization of real noise with a small correlation time for a very large number of applications [1]. White noise also has the advantage that the evolution of the system is Markovian [2], which simplifies considerably the analytical study of the statistical behavior of the system. In the white noise approximation, spatially homogeneous systems with a single-state variable $x(t)$ are often described by the Langevin equation

$$\dot{x}(t)=F(x(t))+G(x(t))\Gamma(t), \tag{1.1}$$

where $F(x)$ and $G(x)$ are deterministic functions that can depend explicitly on t as well, and $\Gamma(t)$ is Gaussian white noise. We are interested in the case of multiplicative noise, i.e., $G(x)$ is not constant and the effect of the noise depends on the state of the system. As is well known, in the case of multiplicative white noise, the Langevin equation (1.1) as written is meaningless until an appropriate interpretation for the integral of the noise term has been adopted [1–3]. To do so, we must specify the parameter λ $(0\leqslant\lambda\leqslant1)$ that determines the points of time at which $G(x(t))$ is evaluated in the corresponding integral sum. Only then can the Langevin equation be properly integrated to obtain sample paths for the stochastic evolution of the system, and a stochastically equivalent Fokker-Planck equation can be associated with Eq. (1.1). Most commonly, the values $\lambda=0$ and $\lambda=1/2$, corresponding to the Ito [4] and Stratonovich [5] interpretation of Eq. (1.1), respectively, are chosen. However, other choices are possible, and Klimontovich [6], for example, has discussed the choice $\lambda=1$, the so-called kinetic form of the Langevin and Fokker-Planck equations. Equation (1.1) with an appropriately chosen value for λ represents a valuable tool for modeling a great variety of phenomena and processes, including stochastic resonance [7], noise-induced transitions [1], resonant activation [8], and directed transport [9], to name a few.

As was recently realized, some applications require that the fluctuating environment of the system be modeled by two, or more, cross-correlated Gaussian white noises. Usually, a multiplicative and an additive noise are considered. Studies of the single-mode laser [10], of noise-induced transport of Brownian particles [11] and quantum particles [12], of the mean first passage time over a fluctuating potential barrier [13], of the activation rate [14], of stochastic resonance [15], and of stochastic systems with colored correlation between white noise and colored noise [16] show that the cross-correlation between those noises plays a significant role. In order to describe the fluctuation effects in systems that are subjected to the action of two white noises, we present a model whose dimensionless state $x(t)$ evolves according to the Langevin equation

$$\dot{x}(t)=f(x(t))+\sum_{i=1}^{2} g_i(x(t))\gamma_i(t). \tag{1.2}$$

To capture the broadest variety of applications, we assume that each Gaussian white noise $\gamma_i(t)$ is characterized by its own parameter λ_i $(0\leqslant\lambda_i\leqslant1)$. The noises have zero mean, and their correlation functions are defined as follows:

$$\langle\gamma_i(t)\gamma_j(t')\rangle=2\Delta_{ij}\delta(t-t'). \tag{1.3}$$

Here $\langle\cdot\rangle$ denotes averaging with respect to the noises $\gamma_i(t)$, $\Delta_{11}\equiv\Delta_1(\geqslant0)$ and $\Delta_{22}\equiv\Delta_2(\geqslant0)$ are the intensities of the noises $\gamma_1(t)$ and $\gamma_2(t)$, respectively, $\Delta_{12}=\Delta_{21}\equiv r\sqrt{\Delta_1\Delta_2}$, r is the coefficient of correlation between $\gamma_1(t)$ and $\gamma_2(t)$, and $\delta(t)$ is the Dirac δ function.

In Ref. [17], the authors study different nonlinear systems with two noises and show that the additive noise, correlated or uncorrelated with the multiplicative noise, can induce

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nonequilibrium transitions. In this paper, we use Eq. (1.2) to study the possibility of nonequilibrium transitions induced by the cross-correlation of white noises. More precisely, we are interested in the situation when each noise separately does not induce transitions, but together they do, due to their cross-correlation. According to Ref. [1], a noise-induced transition occurs at $r=r_{cr}$, if the number of local maxima of the stationary probability distribution function $P_{st}(x)$ of $x(t)$ is different for $r=r_{cr}-0$ and for $r=r_{cr}+0$. To calculate $P_{st}(x)$, we need to obtain the Fokker-Planck equation for the time-dependent probability distribution function of $x(t)$. In the particular case in which Eq. (1.2) is interpreted as a Stratonovich equation, the corresponding Fokker-Planck equation was found earlier [18–20], however by methods that cannot be directly extended to the general case of arbitrary λ_i. To solve this problem, we develop another approach based on the two-stage averaging of a state-dependent function. Within this framework, we derive the Fokker-Planck equation for arbitrary λ_i and show that Eq. (1.2) can be reduced to a stochastically equivalent Langevin equation of form (1.1). We calculate $P_{st}(x)$ and study in detail the extremal properties of $P_{st}(t)$ for systems with a linear or cubic restoring force $f(x)$ and additive noise $\gamma_2(t)$. Finally, we show explicitly that such systems can exhibit nonequilibrium transitions induced by the cross-correlation of the noises.

The paper is organized as follows. In Sec. II, we derive the Fokker-Planck equation which corresponds to Eq. (1.2), and find an equation of form (1.1) that is equivalent to Eq. (1.2). In Sec. III, we obtain the stationary distribution of the solution of Eq. (1.2), and consider three specific examples of Eq. (1.2) that show that the cross-correlation of noises can induce nonequilibrium transitions. We summarize our results in Sec. IV.

II. THE FOKKER-PLANCK EQUATION

To obtain the Fokker-Planck equation, we consider Eq. (1.2) to be the result of the mean-square limit $\tau\rightarrow 0$ of the implicit difference scheme

$$\delta x = f(x(t))\tau + \sum_{i=1}^{2} g_i(x(t_i))\delta W_i . \tag{2.1}$$

Here $\delta x = x(t+\tau)-x(t)$, $t_i = t+\lambda_i\tau$, and $\delta W_i = W_i(t+\tau) - W_i(t)$ are the increments of Wiener processes $W_i(t)$ satisfying the conditions

$$\overline{\delta W_i}=0, \quad \overline{\delta W_i \delta W_j} = 2\Delta_{ij}\tau. \tag{2.2}$$

The overbar denotes averaging with respect to the increments of Wiener processes. Note that the allowed values of r are restricted by the inequality $|r|\leqslant 1$, because $\overline{(\delta W_1 \pm \delta W_2)^2} = 2\tau(\Delta_1+\Delta_2 \pm 2r\sqrt{\Delta_1\Delta_2}) \geqslant 0$.

Since $\delta W_i \propto \tau^{1/2}$ and the equality $x(t_i) = x(t)+\lambda_i \delta x$ is accurate to first order in δx, we obtain from Eq. (2.1) the formula

$$\delta x = f(x(t))\tau + \sum_{i=1}^{2} g_i(x(t))\delta W_i + \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_i g_i'(x(t)) g_j(x(t))\delta W_i \delta W_j , \tag{2.3}$$

which is accurate to first order in τ. (Here and below, the prime denotes the derivative with respect to the argument of the function.) Next, we introduce a doubly differentiable function $u(x)$ and calculate the difference $\delta u = u(x(t+\tau)) - u(x(t))$. Substituting Eq. (2.3) into the approximate formula $\delta u = \delta x u'(x(t)) + 1/2(\delta x)^2 u''(x(t))$, we find with the same accuracy as Eq. (2.3)

$$\begin{aligned}\delta u &= u'(x(t)) f(x(t))\tau + u'(x(t))\sum_{i=1}^{2} g_i(x(t))\delta W_i \\ &+ u'(x(t))\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_i g_i'(x(t)) g_j(x(t))\delta W_i\delta W_j \\ &+ \frac{1}{2}u''(x(t))\sum_{i=1}^{2}\sum_{j=1}^{2} g_i(x(t)) g_j(x(t))\delta W_i \delta W_j .\end{aligned} \tag{2.4}$$

The next step consists in averaging Eq. (2.4). The result is written as $\langle \delta u\rangle = \langle \cdot \rangle$, where the dot denotes the right-hand side of Eq. (2.4). Let $P(x,t)$ be the probability density that $x(t)=x$. Then

$$\langle u(x(t))\rangle = \langle u(x)\rangle_{P(x,t)} \equiv \int_{-\infty}^{\infty} dx u(x) P(x,t), \tag{2.5}$$

and therefore

$$\langle \delta u\rangle = \tau\int_{-\infty}^{\infty} dx\, u(x)\frac{\partial}{\partial t}P(x,t), \tag{2.6}$$

as $\tau\rightarrow 0$. Further, since $\langle\cdot\rangle$ can be represented in the form of a two-stage averaging, i.e., $\langle\cdot\rangle = \langle\overline{\cdot}\rangle_{P(x,t)}$, Eqs. (2.2) and (2.4) yield

$$\begin{aligned}\langle\cdot\rangle = \tau\Big\langle u'(x)f(x) &+ 2u'(x)\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_i\Delta_{ij}g_i'(x)g_j(x) \\ &+ u''(x)\sum_{i=1}^{2}\sum_{j=1}^{2}\Delta_{ij}g_i(x)g_j(x)\Big\rangle_{P(x,t)} .\end{aligned} \tag{2.7}$$

Using the integral representation for mean values, integrating by parts, and assuming as usual natural boundary conditions, i.e., the flow of probability vanishes at infinity, we find

$$\begin{aligned}\langle\cdot\rangle = \tau\int_{-\infty}^{\infty} dx u(x)\Big\{ &-\frac{\partial}{\partial x}[f(x)+h(x)]P(x,t) \\ &+ \frac{\partial^2}{\partial x^2} d(x)P(x,t)\Big\},\end{aligned} \tag{2.8}$$

where

$$h(x)=2\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_i\Delta_{ij}g_i'(x)g_j(x), \tag{2.9}$$

$$d(x)=\sum_{i=1}^{2}\sum_{j=1}^{2}\Delta_{ij}g_i(x)g_j(x). \tag{2.10}$$

Finally, comparing the right-hand sides of Eqs. (2.6) and (2.8) and taking into account that $u(x)$ is an arbitrary function, we obtain the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}[f(x)+h(x)]P(x,t)+\frac{\partial^2}{\partial x^2}d(x)P(x,t), \tag{2.11}$$

which is stochastically equivalent to Eq. (1.2). Using our approach to calculate the conditional average $\langle u(x(t))\rangle|_{x(t_0)=x_0}$ $(t_0\leqslant t)$, we confirm that the conditional probability density $P(x,t|x_0,t_0)$ satisfies the same Fokker-Planck equation (2.11). We emphasize that this equation is also valid in the general case when the functions $f(x)$ and $g_i(x)$ depend explicitly on t.

Our results show that Eq. (1.2) defines a Markovian diffusion process with a drift coefficient equal to $f(x)+h(x)$ and a diffusion coefficient equal to $2d(x)$. The eigenvalues

$$\kappa_{1,2}=\frac{\Delta_1+\Delta_2}{2}\pm\frac{1}{2}\sqrt{(\Delta_1-\Delta_2)^2+4r^2\Delta_1\Delta_2} \tag{2.12}$$

of the matrix $[\Delta_{ij}]$ are non-negative, which implies that Eq. (2.10) is a non-negative definite quadratic form, i.e., $d(x)\geqslant 0$. The completely degenerate case $d(x)\equiv 0$ occurs if (1) $g_1(x)=g_2(x)\equiv 0$; (2) $\Delta_1=\Delta_2=0$; (3) $\Delta_1=0$, $\Delta_2>0$, and $g_2(x)\equiv 0$ [or $\Delta_2=0$, $\Delta_1>0$, and $g_1(x)\equiv 0$]; (4) $r=-1$, $\Delta_1=\Delta_2>0$, and $g_1(x)=g_2(x)$ [or $r=1$, $\Delta_1=\Delta_2>0$, and $g_1(x)=-g_2(x)$]. Conditions (1)–(3) represent the trivial case in which noises do not affect the system, and condition (4) means that the noises exactly cancel each other, i.e., $\langle[\gamma_1(t)\pm\gamma_2(t)]^2\rangle=0$ if $r=\mp 1$. In all other cases, $d(x)\not\equiv 0$.

The Fokker-Planck equation associated with Eq. (1.1), in which the noise $\Gamma(t)$ is characterized by the correlation function $\langle\Gamma(t)\Gamma(t')\rangle=2\Delta\delta(t-t')$ [here $\langle\cdot\rangle$ denotes averaging with respect to the noise $\Gamma(t)$] and by the parameter λ, has the form [21]

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}[F(x)+H(x)]P(x,t)+\frac{\partial^2}{\partial x^2}D(x)P(x,t), \tag{2.13}$$

where $H(x)=2\lambda\Delta G'(x)G(x)$ and $2D(x)=2\Delta G^2(x)$ are the noise-induced drift and the diffusion coefficient, respectively. Comparing Eq. (2.11) with Eq. (2.13), we conclude that Eq. (1.1) is stochastically equivalent to Eq. (1.2) (we assume the same initial condition for both equations), if

$$G(x)=\sqrt{d(x)/\Delta}, \tag{2.14}$$

$$F(x)=f(x)+h(x)-\lambda d'(x). \tag{2.15}$$

Note that for systems with one noise the relation $H(x)=\lambda D'(x)$ always holds, i.e., $H(x)\propto D'(x)$. For systems with two noises it is replaced by the more general relation

$$h(x)=\sum_{i=1}^{2}\lambda_i g_i'(x)\frac{\partial d(x)}{\partial g_i(x)}. \tag{2.16}$$

Therefore, $h(x)\propto d'(x)$ holds only for specific cases; for example, if $g_1(x)=\text{const}$, if $\lambda_1=\lambda_2$, or if $g_1(x)=g_2(x)$.

III. ANALYSIS OF THE STATIONARY DISTRIBUTION

The stationary solution of the Fokker-Planck equation (2.11) is given by

$$P_{st}(x)=\frac{1}{Zd(x)}\exp\left[\int_\gamma^x dy\frac{f(y)+h(y)}{d(y)}\right], \tag{3.1}$$

if the normalizing factor

$$Z=\int_\alpha^\beta dx\frac{1}{d(x)}\exp\left[\int_\gamma^x dy\frac{f(y)+h(y)}{d(y)}\right] \tag{3.2}$$

$[x(t)\in(\alpha,\beta),\alpha\leqslant\gamma\leqslant\beta]$ exists. Assuming that $d(x)>0$, we write the equation $P_{st}'(x)=0$, which defines the location of the extrema of $P_{st}(x)$, in the form

$$f(x)+h(x)-d'(x)=0. \tag{3.3}$$

As mentioned in the Introduction, we seek transitions that are induced by the cross-correlation of the noises. This means that for $r=0$, Eq. (3.3) has the same number of roots as equation $f(x)=0$, and for $r=r_{cr}$ that number changes. Next we demonstrate explicitly that such transitions indeed occur even in relatively simple systems with $g_2(x)=1$, i.e., when the noise $\gamma_2(t)$ is additive.

A. First example

As a first example, we consider a system with a linear restoring force $f(x)=-ax$ $(a>0)$ and

$$g_1(x)=b\frac{x^2}{1+x^2}\quad(b>0). \tag{3.4}$$

For this system $\alpha=-\infty$ and $\beta=\infty$. According to Eqs. (2.9) and (2.10), $h(x)=\lambda_1 d'(x)$,

$$d(x)=\Delta_1 b^2\left[\left(\frac{x^2}{1+x^2}\right)^2+2r\nu\frac{x^2}{1+x^2}+\nu^2\right], \tag{3.5}$$

and so Eq. (3.3) is reduced to

$$x[z^3+\eta(1+r\nu)z-\eta]=0, \tag{3.6}$$

where $z=1+x^2$ and

$$\eta=\frac{4(1-\lambda_1)\Delta_1 b^2}{a},\quad \nu=\frac{1}{b}\sqrt{\frac{\Delta_2}{\Delta_1}}. \tag{3.7}$$

If $r=0$, then $z^3+\eta z-\eta\geqslant 1$, and Eq. (3.6), just as equation $f(x)=0$, has the unique real root $x=0$, which corresponds to the maximum of $P_{st}(x)$. This means that uncorrelated noises do not change the unimodal character of $P_{st}(x)$, i.e., noise-induced transitions do not occur. They can occur if the cubic equation

$$z^3+\eta(1+r\nu)z-\eta=0 \tag{3.8}$$

has real roots which satisfy the condition $z>1$. Simple analysis shows that among the three roots z_n ($n=\overline{1,3}$) of Eq. (3.8) only one root, say z_3, satisfies that condition. Indeed, as is well known (see, for example, Ref. [22]), if the parameter

$$Q=\frac{\eta^2}{4}+\frac{\eta^3(1+r\nu)^3}{27} \tag{3.9}$$

is positive, then Eq. (3.8) has two complex-conjugate roots z_1 and z_2, and one real root z_3. It is not difficult to verify that $z_3>1$, only if $1+r\eta\nu<0$. Further, at $Q=0$ all roots of Eq. (3.8) are real, and at last two roots, say z_1 and z_2, are equal. Using the Vieta theorem, which expresses the coefficients of a polynomial in terms of its roots, we can write the relations

$$z_1+z_2+z_3=0,\quad z_1z_2z_3=\eta, \tag{3.10}$$

which show that in this case roots z_1 and z_2 are negative and must be rejected. Finally, for $Q<0$ all roots are real and different, and for the same reason as in the previous case, roots z_1 and z_2 must be rejected also.

Thus, if $r>r_{cr}=-1/\eta\nu$, then Eq. (3.6) has the unique real root $x=0$, and the stationary probability distribution function

$$P_{st}(x)=C\frac{\exp\left[-q\int_0^{x^2}\frac{dz}{\left(\frac{z}{1+z}\right)^2+2r\nu\frac{z}{1+z}+\nu^2}\right]}{\left[\left(\frac{x^2}{1+x^2}\right)^2+2r\nu\frac{x^2}{1+x^2}+\nu^2\right]^{1-\lambda_1}} \tag{3.11}$$

[C is the normalizing constant, $q=2(1-\lambda_1)/\eta$] is unimodal with the global maximum located at the point $x=0$ (see Fig. 1). For $r<r_{cr}$ (this condition can be fulfilled if the inequality $|r_{cr}|<1$ holds), Eq. (3.6) has three roots $x=0$, $x=\tilde{x}$, and $x=-\tilde{x}$, where $\tilde{x}=\sqrt{z_3-1}$. In this case, the point $x=0$ corresponds to a local minimum of the function $P_{st}(x)$, and the points $x=\tilde{x}$ and $x=-\tilde{x}$ to local maxima of equal height, i.e., $P_{st}(x)$ is bimodal (see Fig. 2). Therefore at $r=r_{cr}$, a unimodal-bimodal transition occurs. Note also that the distance $2\tilde{x}$ between the local maxima of $P_{st}(x)$ grows as r decreases. In particular, if $r=r_{cr}-\epsilon$ and $\epsilon/|r_{cr}|\ll 1$, then $\tilde{x}=\sqrt{\epsilon\eta\nu/(2+\eta)}$ and $\tilde{x}=0$ at $r=r_{cr}$.

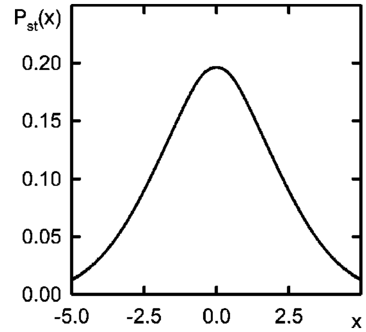

FIG. 1. Plot of $P_{st}(x)$ vs x for Eq. (3.11); $r=0$ and $\eta=2$, $\nu=2$, $\lambda_1=0.5$ ($r_{cr}=-0.25$).

B. Second example

As another example consider a system with a cubic restoring force $f(x)=-2ax^3$ ($a>0$, the factor 2 is introduced for convenience) and

$$g_1(x)=b\frac{x^4}{1+x^4}\quad (b>0). \tag{3.12}$$

For this system $h(x)=\lambda_1 d'(x)$,

$$d(x)=\Delta_1 b^2\left[\left(\frac{x^4}{1+x^4}\right)^2+2r\nu\frac{x^4}{1+x^4}+\nu^2\right], \tag{3.13}$$

and the stationary probability distribution function is given by

$$P_{st}(x)=C\frac{\exp\left[-q\int_0^{x^4}\frac{dz}{\left(\frac{z}{1+z}\right)^2+2r\nu\frac{z}{1+z}+\nu^2}\right]}{\left[\left(\frac{x^4}{1+x^4}\right)^2+2r\nu\frac{x^4}{1+x^4}+\nu^2\right]^{1-\lambda_1}}. \tag{3.14}$$

Since in this case Eq. (3.3) can be reduced to Eq. (3.6) with $z=1+x^4$, we conclude that the form of the probability distribution function (3.14) depends on the control parameter r qualitatively in the same way as in the previous case. Namely, for $r>r_{cr}=-1/\eta\nu$ the function $P_{st}(x)$ is a unimo-

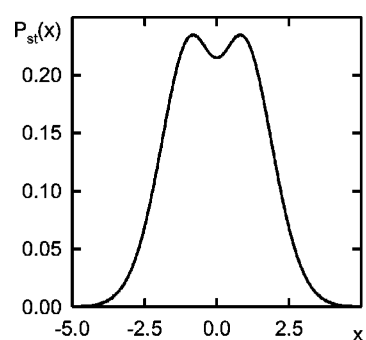

FIG. 2. Plot of $P_{st}(x)$ vs x for Eq. (3.11); $r=-0.9$ and the other parameters have the same values as in Fig. 1.

dal distribution, for $r<r_{cr}$ it is a bimodal distribution, and at $r=r_{cr}$ a unimodal-bimodal transition occurs.

These transitions have the distinctive feature that the global maximum of $P_{st}(x)$ is transformed into the local minimum at the transition point $r=r_{cr}$.

C. Third example

We consider a system with a linear restoring force and with $g_1(x)$ given by Eq. (3.12). In this case, Eqs. (3.1) and (3.13) yield

$$P_{st}(x)=C\frac{\exp\left[-q\int_0^{x^2}\frac{dz}{\left(\frac{z^2}{1+z^2}\right)^2+2r\nu\frac{z^2}{1+z^2}+\nu^2}\right]}{\left[\left(\frac{x^4}{1+x^4}\right)^2+2r\nu\frac{x^4}{1+x^4}+\nu^2\right]^{1-\lambda_1}}, \tag{3.15}$$

and Eq. (3.3) is reduced to

$$x[(1+x^4)^3+2\eta(1+r\nu)x^6+2\eta r\nu x^2]=0. \tag{3.16}$$

According to Eq. (3.16), r_{cr} can be represented as $r_{cr}=-\nu^{-1}\varphi(\eta)$, where $\varphi(\eta)$ is a positive function of η. Numerical analysis shows that if $r>r_{cr}$, then Eq. (3.16) has the unique real root $x=0$, and $P_{st}(x)$ is unimodal as in Fig. 1. If $r<r_{cr}$ (this condition can hold if $|r_{cr}|<1$), then Eq. (3.16) has five roots; and three of them, $x=0$, $x=-\tilde{x}$, and $x=\tilde{x}$ ($\tilde{x}$ grows as r decreases), correspond to the local maxima of $P_{st}(x)$ (see Fig. 3). Thus, at $r=r_{cr}$ a unimodal-trimodal transition occurs. Note that in this case $\tilde{x}\neq 0$ at $r=r_{cr}$, in contrast to the previous examples.

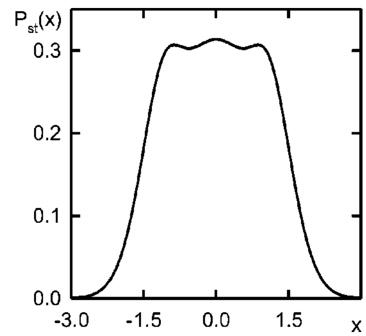

FIG. 3. Plot of $P_{st}(x)$ vs x for Eq. (3.15); $r=-1$ and $\eta=1$, $\nu=2$, $\lambda_1=0.5$ ($r_{cr}\approx-0.88$).

IV. CONCLUSIONS

We have studied the role that the cross-correlation of noises plays in the phenomenon of noise-induced transitions. Starting from a Langevin equation with two multiplicative cross-correlated white noises and adopting a general interpretation of the noise terms, we have derived the corresponding Fokker-Planck equation. For this purpose we have developed a simple approach, which is based on the two-stage averaging of a state-dependent function. Analyzing the Fokker-Planck equation, we have shown that the two-noise Langevin equation can be reduced to a stochastically equivalent one-noise Langevin equation.

Also, we have studied analytically and numerically the influence of the cross-correlation of noises on the stationary solution of the Fokker-Planck equation. For specific cases of systems with linear and cubic restoring forces, we have shown that changing the strength of the cross-correlation can lead to a qualitative change of the stationary probability distribution function. Specifically, in such systems unimodal-bimodal and unimodal-trimodal transitions can occur.

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